Fixed rings: Minimal ring extensions

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Abstract

By comparison to linear algebra, which includes the study of vector spaces, abstract algebra includes the study of groups, rings, and fields. Fields are often encountered in other areas of mathematics, and certain groups and rings are also common, e.g., a vector space is group, and the integers $\mathbb{Z}$ are a ring. Moreover, $\mathbb{Z}$ is a *commutative* ring, meaning $a \cdot b = b \cdot a$ for all integers a and b. We begin with a thorough introduction to groups and commutative rings as well as several examples.

We then define the automorphism group and fixed ring of a commutative ring R. An *automorphism* is an isomorphism $\sigma : R \to R$ that preserves the ring's structure, e.g., $\sigma : \mathbb{C} \to \mathbb{C}$ given by $a + bi \mapsto a - bi$. The collection of these maps forms a group under composition called the *automorphism group*. Now consider a subgroup G of this group. We call the collection of elements of R that are stable under all automorphisms in G the *fixed ring*, denoted R^G. That is, $R^G = \{r \in R \,|\, \sigma(r) = r \text{ for all } \sigma \in G\}$. If S is a subring of R, then S^G is analogously defined.

Inspired by Hilbert's Fourteenth Problem, properties of R inherited by R^G and properties of the extension $R^G \subseteq R$ have been studied extensively. We expand this work by determining properties of the extension $S \subset R$ inherited by $S^G \subseteq R^G$. If there no (proper) intermediate rings (between S and R), then $S \subset R$ is called a *minimal ring extension*. Assuming $S^G \neq R^G$, we show that this property is passed down to $S^G \subset R^G$ for certain groups G.