Infinite groups and primitivity of their group rings

Tsunekazu Nishinaka *
Department of Business Administration
Okayama Shoka University

A ring R is (right) primitive provided it has a faithful irreducible (right) R-module. If non-trivial group G is finite or abelian, then the group algebra KG over a field K cannot be primitive. If G has non-abelian free subgroups, then KG is often primitive. In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

($*$) For each subset M of G consisting of finite number of elements not equal to 1, there exist three distinct elements a, b, c in G such that whenever $x_i \in \{a, b, c\}$ and $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$ for some $g_i \in M$, $x_i = x_{i+1}$ for some i.

We can see that the group algebra KG of a group G over a field K is primitive provided G has a free subgroup with the same cardinality as G and satisfies ($*$). In particular, for every countably infinite group G satisfying ($*$), KG is primitive for any field K. As an application of this theorem, we can see primitivity of group algebras of many kinds of groups with non-abelian free subgroups which includes a recent result; the primitivity of group algebras of one relator groups with torsion.

1 A brief history of the research

Let R be a ring with the identity element. It need not to be commutative. A ring R is right primitive if and only if there exists a faithful irreducible right R-module M_R, where M_R is irreducible provided it has no non-trivial submodules, and M_R is faithful provided the annihilator of M is zero: $ann(M_R) = \{r \in R \mid Mr = 0\} = 0$. The above definition of primitivity is equivalent to the following: A ring R is right primitive if and only if there exists a maximal right ideal ρ which contains no non-trivial ideals of R. A left primitive ring is similarly defined. In what follows, for right primitive, we simply call it primitive. Speaking of a group ring, a right primitive group ring is always left primitive. In this section, we introduce briefly a history of the research to primitivity of group rings.

Since the group ring KG of a non-trivial group G over a field K has always the augmentation ideal which is non-trivial, it cannot be a simple ring. If G is a finite group, then KG is a finite dimensional algebra and so it is never primitive because a finite dimensional algebra is simple provided it is primitive. Moreover,

*Partially supported by Grants-in-Aid for Scientific Research under grant no. 23540063

if a commutative ring is primitive, then it is a field, and therefore if $G \neq 1$ is abelian, then KG is never primitive. Hence primitivity of KG is appeared only in the case that G is non-abelian and non-finite. For the longest time no examples of primitive group rings were known, and it was thought that KG could not be primitive provided $G \neq 1$.

The first example of primitive group rings was offered by Formanek and Snider [7] in 1972, and in 1973 Formanek [6] gave the primitivity of group rings of well-known groups; namely the primitivity of group rings of free products.

Theorem 1.1. (Formanek[6]) *Let G be a free product of non-trivial groups (except $G = \mathbb{Z}_2 * \mathbb{Z}_2$); Then KG is primitive for any field K.*

In particular, if G is a free group then KG is primitive for any field K. After that, many examples of primitive group rings were constructed. In 1978, Domanov [4], Farkas-Passman [5] and Roseblade [17] gave the complete solution for primitivity of group rings of polycyclic-by-finite groups.

Theorem 1.2. (Domanov[4], Farkas-Passman[5],Roseblade[17]) *Let G be a non-trivial polycyclic-by-finite group. Then KG is primitive if and only if $\Delta(G) = 1$ and K is non-absolute, where $\Delta(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ and K is absolute if it is algebraic over a finite field.*

Polycyclic-by-finite groups are belong to the class of noetherian groups, and it is not easy to find a noetherian group which is not polycyclic-by-finite [15]. Therefore almost all other known infinite groups belong to the class of non-noetherian groups. As is well known, if KG is noetherian then G is also noetherian, but the converse is not true generally. A group of the class of finitely generated non-noetherian groups has often non-abelian free subgroups; for instance, a free group, a locally free group, a free product, an amalgamated free product, an HNN-extension, a Fuchsian group, a one relator group, etc. It is known that a free Burnside group is not the case, though. After the result Theorem 1.1 above, primitivity of group rings of known groups which are non-noetherian has been obtained gradually. Theorem 1.1 was generalized to one for amalgamated free products by Balogun in 1989:

Theorem 1.3. ([1, Balogun, '89]) *Let $G = A *_H B$ be the free product of A and B with H amalgamated. If there exist elements $a \in A \setminus H$ and $b \in B \setminus H$ such that $a^2, b^2 \notin H$, $a^{-1}Ha \cap H = 1$ and $b^{-1}Hb \cap H = 1$, then KG is primitive for any field K.*

In 1997, the primitivity of semigroup algebras of free products was given by Chaudhry, Crabb and McGregor [2].

The primitivity of a group ring of a free group F extended to one for the ascending HNN extension $G = F_\varphi$ of a free group F; for the case of $|F| = \aleph_0$ in 2007 and for the case of arbitrary cardinality of F in 2011:

Theorem 1.4. ([13, Nishinaka, '07], [14, Nishinaka, '11]) *Let F be a non-abelian free group, and $G = F_\varphi$ the ascending HNN extension of F determined by φ. Then the following are equivalent:*

(1) *KG is primitive for a field K.*

(2) *$|K| \le |F|$ or G is not virtually the direct product $F \times \mathbb{Z}$.*

(3) *$|K| \le |F|$ or $\triangle(G) = 1$.*

In particular, if G is a strictly ascending HNN extension, that is, $\varphi(F) \neq F$, then KG is primitive for any field K.

Moreover, the primitivity of group rings of free groups extended to one for locally free groups:

Theorem 1.5. ([14, Nishinaka, '11]) *Let G be a non-abelian locally free group which has a free subgroup whose cardinality is the same as that of G itself. If K is a field then KG is primitive.*

In particular, every group ring of the union of an ascending sequence of non-abelian free groups over a field is primitive, and so every group ring of a countable non-abelian locally free group over a field is primitive.

Now, there is no viable conjecture as to when KG is primitive for arbitrary groups. There exists a non-primitive KG for any field K even in the case that KG is semiprimitive and $\Delta(G) = 1$ (See [3]).

2 Group algebras of groups with free subgroups

In the present note, we focus on a local property which is often satisfied by groups with non-abelian free subgroups:

(∗) For each subset M of G consisting of finite number of elements not equal to 1, there exist three distinct elements a, b, c in G such that whenever $x_i \in \{a, b, c\}$ and $(x_1^{-1} g_1 x_1) \cdots (x_m^{-1} g_m x_m) = 1$ for some $g_i \in M$, $x_i = x_{i+1}$ for some i.

We can see that if G is countably infinite group and satisfies (∗), then KG is primitive for any field K. More generally, we can get the following theorem:

Theorem 2.1. *Let G be a non-trivial group which has a free subgroup whose cardinality is the same as that of G. Suppose that G satisfies the condition (∗). If R is a domain with $|R| \le |G|$, then the group ring RG of G over R is primitive.*

In particular, the group algebra KG is primitive for any field K.

As an application of the theorem, we give the primitivity of group algebras of one relator groups with torsion:

Theorem 2.2. *If G is a non-cyclic one relator group with torsion, then KG is primitive for any field K.*

One of the main method to prove Theorem 2.1 is a graph theoretic method which is called SR-graph theory.

3 SR-graph theory

Let $\mathcal{G} = (V, E)$ denote a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. A finite sequence $v_0e_1v_1 \cdots e_pv_p$ whose terms are alternately elements e_q's in E and v_q's in V is called a path of length p in $\mathcal{G}$ if $v_q \neq v_{q'}$ for any $q, q' \in \{0, 1, \cdots, p\}$ with $q \neq q'$; it is often simply denoted by $v_0v_1 \cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's ($1 \leq i \leq m$) for some $m > 0$ such that $v, w \in V$ are connected if and only if both v and w belong to the same set C_i. The subgraph (C_i, E_i) of $\mathcal{G}$ generated by C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components. For $v \in V$, we denote by $C(v)$ the component of $\mathcal{G}$ which contains the vertex v.

We define a graph which has two distinct edge sets E and F on the same vertex set V. We call such a triple (V, E, F) an SR-graph provided that $(V, E \cup F)$ is a simple graph (i.e. a finite undirected graph which has no multiple edges or loops) and every component of the graph (V, E) is a complete graph (see Fig 1 and Fig 2). That is, we define an SR-graph as follows:

Definition 3.1. *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{H}$ and v itself: $U(v) = \{w \in V \mid vw \in F\} \cup \{v\}$. A triple (V, E, F) is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions*:

(SR1) *For any $v \in V$, $C(v) \cap U(v) = \{v\}$.*
(SR2) *Every component of $\mathcal{G}$ is a complete graph.*

If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph (V, E, F) is called a proper SR-graph.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $\mathcal{S} = (V, E, F)$ be an SR-graph. We say $\mathcal{S}$ is connected if the graph $(V, E \cup F)$ is connected.

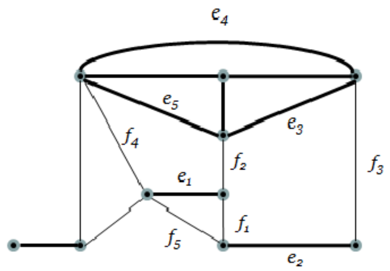

Fig 1. An example of an SR-graph: bold solid lines are edges in E and normal solid lines are edges in F. Sequences $(e_1, f_1, e_2, f_3, e_4, f_4)$, $(e_1, f_2, e_3, f_3, e_2, f_5)$ and (e_1, f_2, e_5, f_4) are SR-cycles.

Fig 2. Prohibits: It is not allowed to exist the above subgraph in an SR-graph.

Definition 3.2. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1 w_1 v_2 w_2, \cdots, v_p w_p v_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length p in $\mathcal{S}$ if either $e_q = v_q w_q \in E$ and $f_q = w_q v_{q+1} \in F$ or $f_q = v_q w_q \in F$ and $e_q = w_q v_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively. If, in addition, it is a cycle in $(V, E \cup F)$; namely, $v_{p+1} = v_1$, then it is an SR-cycle of length p in $\mathcal{S}$.*

To prove Theorem 2.1, we use some results for SR-graphs and apply them to the Formanek's method. We can give Formanek's method, as follows:

Proposition 3.3. (See [6]) *Let RG be the group ring of a group G over a ring R with identity. If for each non-zero $a \in RG$, there exists an element $\varepsilon(a)$ in the ideal $RGaRG$ generated by a such that the right ideal $\rho = \sum_{a \in RG \setminus \{0\}} (\varepsilon(a) + 1) RG$ is proper; namely, $\rho \neq RG$, then RG is primitive.*

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper. Now, ρ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since ρ is generated by the elements of form $(\varepsilon(a) + 1)$ with $a \neq 0$, r has the presentation, $r = \sum_{(a,b) \in \Pi} (\varepsilon(a) + 1) b$, where Π is a subset which consists of finite number of elements of $RG \times RG$ both of whose components are non-zero. Moreover, $\varepsilon(a)$ and b are linear combinations of elements of G, and so we have

$$r = \sum_{(a,b) \in \Pi} \sum_{g \in S_a, h \in T_b} (\alpha_g \beta_h g h + \beta_h h), \tag{1}$$

where S_a and T_b are the support of $\varepsilon(a)$ and b respectively and both α_g and β_h are elements in K. In the above presentation (1), if there exists gh such that $gh \neq 1$ and does not coincide with the other $g'h'$'s and h''s, then $r \neq 1$ holds. Strictly speaking: Let $\Omega_{ab} = S_a \times T_b$. If there exist $(a, b) \in \Pi$ and (g, h) in Ω_{ab} with $gh \neq 1$ such that $gh \neq g'h'$ and $gh \neq h'$ for any $(c, d) \in \Pi$ and for any (g', h') in Ω_{cd} with $(g', h') \neq (g, h)$, then $r \neq 1$ holds.

On the contrary, if $r = 1$, then for each gh in (1) with $gh \neq 1$, there exists another $g'h'$ or h' in (1) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist (g_{2i-1}, h_i) and (g_{2i}, h_{i+1}) $(i = 1, \cdots, m)$ in $V = \bigcup_{(a,b)\in\Pi} \Omega_{ab} \cup T_b$ such that the following equations hold:

$$\begin{array}{llll} g_1h_1 = & g_2h_2, & & \\ & g_3h_2 = & g_4h_3, & \\ & & \ddots & \\ & & & g_{2m-1}h_m = g_{2m}h_{m+1} \quad \text{and} \quad h_{m+1} = h_1. \end{array} \tag{2}$$

Eliminating h_i's in the above, we can see that these equations imply the equation $g_1^{-1}g_2 \cdots g_{2m-1}^{-1}g_{2m} = 1$. If we can choose $\varepsilon(a)$'s so that their supports g_i's never satisfy such an equation, then we can prove that $r \neq 1$ holds by contradiction. We need therefore only to see when supports g's of $\varepsilon(a)$'s satisfy equations as described in (2).

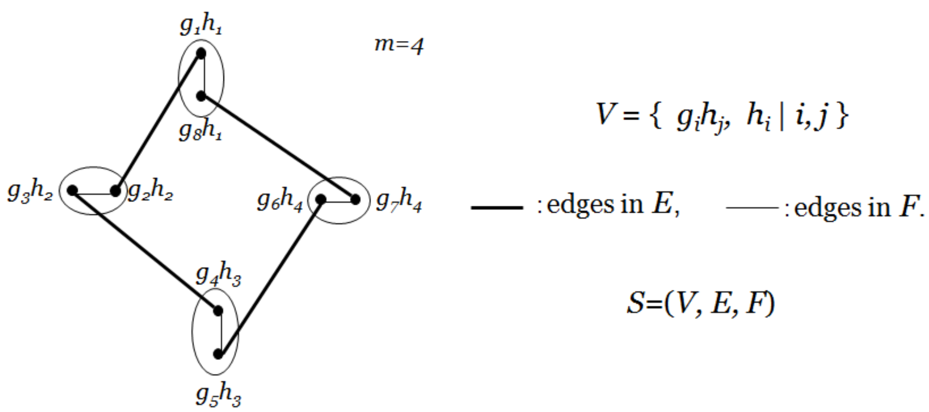

Fig 3. Equations as described in (2) for m=4.

Roughly speaking, we regard V above as the set of vertices and for $v = (g, h)$ and $w = (g', h')$ in V, we take an element vw as an edge in E provided $gh = g'h'$ in G, and take vw as an edge in F provided $g \neq g'$ and $h = h'$ in G (see Fig 3). In this situation, if there exists an SR-cycle $v_1w_1v_2w_2, \cdots, v_pw_pv_1$ in the SR-graph (V, E, F) whose adjacent terms are alternately elements v_iw_i in E and w_iv_{i+1} in F, then there exist (g_i, h_j)'s in V satisfying the desired equations as described in (2). Thus the problem can be reduced to find an SR-cycle in a given SR-graph.

By making use of graph theoretic considerations, we can prove the following theorems:

Theorem 3.4. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and let ω_E and ω_F be, respectively, the number of components of $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$. Suppose that every component of $\mathcal{H} = (V, F)$ is a complete graph and $\mathcal{S}$ is connected. Then $\mathcal{S}$ has an SR-cycle if and only if $\omega_E + \omega_F < |V| + 1$.*

In particular, if $\mathcal{S}$ is proper and $\alpha \leq \gamma$ then $\mathcal{S}$ has an SR-cycle.

Theorem 3.5. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $\mathfrak{C}(V) = \{V_1, \cdots, V_n\}$ with $n > 0$. Suppose that every component $\mathcal{H}_i = (V_i, F_i)$ of $\mathcal{H}$ is a complete k-partite graph with $k > 1$, where k is depend on $\mathcal{H}_i$. If $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in \{1, \cdots, n\}$ and $|I_{\mathcal{G}}(V)| \leq n$ then $\mathcal{S}$ has an SR-cycle.*

4 Proof of Theorem 2.1

Let G be a group and $M_1, \cdots, M_n$ non-empty subsets of G which do not include the identity element. We say $M_1, \cdots, M_n$ are mutually reduced in G if for each finite elements $g_1, \cdots, g_m$ in the union of M_i's, $g_1 \cdots g_m = 1$ implies both g_i and g_{i+1} are in the same M_j for some i and j. If $M_1 = \{x_1^{\pm 1}\}, \cdots, M_m = \{x_m^{\pm 1}\}$ and they are mutually reduced, then we say simply $x_1, \cdots, x_m$ are mutually reduced.

In this section, we shall prove Theorem 2.1 after preparing three lemmas.

Lemma 4.1. (See [16, Theorem 2]) *Let K' be a field and G a group. If $\triangle(G)$ is trivial and $K'G$ is primitive, then for any field extension K of K', KG is primitive.*

By making use of Theorem 3.4 and Theorem 3.5, we can get the next two lemmas:

Lemma 4.2. *Let G be a non-trivial group, $m > 0$ and $n > 0$. For non-trivial distinct elements f_{ij}'s ($i = 1, 2, 3,\ j = 1, \cdots, m$) in G and for distinct elements g_i's ($i = 1, \cdots, n$) in G, we set*

$$\begin{aligned}
S &= \textstyle\bigcup_{i=1}^{3} S_i, \ \text{where } S_i = \{f_{ij} \mid 1 \leq j \leq m\},\\
T &= \{g_i \mid 1 \leq i \leq n\},\\
V &= S \times T,\\
M_i &= \{f_{ij}^{\pm 1},\ f_{ij}^{-1} f_{ik} \mid j, k = 1, 2, \cdots, m,\ j \neq k\}\ (i = 1, 2, 3),\\
I &= \{(f, g) \in V \mid fg \neq f'g' \ \text{for any } (f', g') \in V \text{ with } (f', g') \neq (f, g)\}.
\end{aligned}$$

Then if M_1, M_2 and M_3 are mutually reduced, then $|I| > n$.

Lemma 4.3. *Let G be a non-trivial group and $n > 0$. For each $i = 1, 2, \cdots, n$, let $f_{i1}, \cdots, f_{im_i}$ be distinct $m_i > 0$ elements of G; $f_{ip} \neq f_{iq}$ for $p \neq q$, and let x_{ij} $(1 \leq i \leq n, 1 \leq j \leq 3)$ be distinct elements in G. we set*

$$\begin{array}{lll} S & = \bigcup_{i=1}^{3} S_i, & \text{where } S_i = \{f_{ij} \mid 1 \leq j \leq m_i\}, \\ X & = \bigcup_{i=1}^{n} X_i, & \text{where } X_i = \{x_{ij} \mid 1 \leq j \leq 3\}, \\ V & = \bigcup_{i=1}^{n} V_i, & \text{where } V_i = X_i \times S_i, \\ I & = \{(x, f) \in V \mid xf \neq x'f' & \text{for any } (x', f') \in V \text{ with } (x', f') \neq (x, f)\}. \end{array}$$

If x_{ij}'s are mutually reduced elements, then $|I| > m$, where $m = m_1 + \cdots + m_n$.

Proof of Theorem 2.1. Let B be the basis of a free subgroup of G whose cardinality is the same as that of G. Then we may assume that the cardinality of B is also same as G, that is, $|B| = |G|$. In addition, since $|R| \leq |G|$, we have that $|B| = |RG|$. We can divide B into three subsets B_1, B_2 and B_3 each of whose cardinality is $|B|$. It is then obvious that the elements in B are mutually reduced. Let φ be a bijection from B to $RG \setminus \{0\}$ and σ_s a bijection from B to B_s, $s = 1, 2, 3$.

For $b \in B$, let $\varphi(b) = \sum_{f \in F_b} \alpha_f f$, where $\alpha_f \in R$ and F_b is the support of $\varphi(b)$. We set

$$M_b = \{f^{\pm 1},\ f^{-1}f' \mid f, f' \in F_b, f \neq f'\}.$$

Since G satisfies the condition $(*)$, there exist $x_{b1}, x_{b2}, x_{b3} \in G$ such that $M_b^{x_{bt}} = \{x_{bt}^{-1} f^{\pm 1} x_{bt},\ x_{bt}^{-1} f^{-1} f' x_{bt} \mid f, f' \in F_b,\ f \neq f'\}$ $(t = 1, 2, 3)$ are mutually reduced. We here define $\varepsilon(b)$ and $\varepsilon^1(b)$ by

$$\varepsilon(b) = \sum_{s=1}^{3} \sum_{t=1}^{3} \sigma_s(b) x_{bt}^{-1} \varphi(b) x_{bt} \quad \text{and} \quad \varepsilon^1(b) = \varepsilon(b) + 1. \tag{3}$$

Note that $\varepsilon(b)$ is an element in the ideal of RG generated by $\varphi(b)$. Let $\rho = \sum_{b \in B} \varepsilon^1(b) RG$ be the right ideal generated by $\varepsilon^1(b)$ for all $b \in B$. If $w \in \rho$, then we can express w by

$$w = \sum_{b \in A} \varepsilon^1(b) u_b = \sum_{b \in A} (\varepsilon(b) u_b + u_b) \tag{4}$$

for some non-empty finite subsets A of B and u_b in RG. In view of Proposition 3.3, in order to prove that RG is primitive, we need only show that ρ is proper; $\rho \neq RG$. To do this, it suffices to show that $w \neq 1$.

Let $u_b = \sum_{h \in H_b} \beta_h h$, where H_b is the support of u_b. Substituting this and $\varphi(b) = \sum_{f \in F_b} \alpha_f f$ into (3), we obtain the following expression of $\varepsilon(b) u_b$:

$$\varepsilon(b) u_b = \sum_{s=1}^{3} \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_{bs} x_{bt}^{-1} f x_{bt} h, \text{ where } y_{bs} = \sigma_s(b). \tag{5}$$

In what follows, for the sake of convenience, we represent $y_{bs}x_{bt}^{-1}fx_{bt}h$ by $y_sx_t^{-1}fx_th$, and we note that y_s and x_t are depend on $b \in B$. For $s = 1, 2, 3$, we here set

$$E_{bs} = \sum_{t=1}^{3} \sum_{f \in F_b} \sum_{h \in H_b} \alpha_f \beta_h y_s \xi(x_t, f, h), \text{ where } \xi(x_t, f, h) = x_t^{-1} f x_t h. \qquad (6)$$

That is, $\varepsilon(b)u_b = E_{b1} + E_{b2} + E_{b3}$. We can see that there exist more than $|H_b|$ isolated elements in the expression (6) of E_{bs} for each $s = 1, 2, 3$. Strictly speaking, if we set $X_b = \{x_1, x_2, x_3\}$, $\Gamma_b = X_b \times F_b \times H_b$ and

$$\begin{aligned} I_s = \{(x_t, f, h) \quad & | \ (x_t, f, h) \in \Gamma_b,\ \xi(x_t, f, h) \neq \xi(x_p, f', h') \\ & \text{for any } (x_p, f', h') \in \Gamma_b \text{ with } (x_p, f', h') \neq (x_t, f, h)\}, \end{aligned}$$

then $|I_s| > |H_b|$. In fact, since $M_b^{x_{bt}}$ $(t = 1, 2, 3)$ are mutually reduced, it follows from lemma 4.2 that $|I_s| > |H_b|$.

Now, we shall see that $w \neq 1$ holds, where w as in (4). In (4), we set that $w_1 = \sum_{b \in A} \varepsilon(b)u_b$ and $w_2 = \sum_{b \in A} u_b$. We have then that

$$w_1 = \sum_{b \in A} \sum_{s=1}^{3} E_{bs} \quad \text{and} \quad w = w_1 + w_2.$$

Let $Supp(E_{bs})$ be the support of E_{bs} and $m_b = |Supp(E_{b1})|$. We should note that $|Supp(E_{bs})| = m_b$ for all $s = 1, 2, 3$. It is obvious that $m_b \geq |I_s|$, and so $m_b > |H_b|$ by the above. Since y_{bs} $(b \in A, 1 \leq s \leq 3)$ are mutually reduced, by virtue of Lemma 4.3, we have $|Supp(w_1)| > \sum_{b \in A} m_b$. Moreover we have that

$$\begin{aligned} |Supp(w)| \ & \geq |Supp(w_1)| - |Supp(w_2)| \\ & > \sum_{b \in A} m_b - \sum_{b \in A} |H_b| \\ & > 0, \end{aligned}$$

which implies $|Supp(w)| \geq 2$. In particular, $w \neq 1$. We have thus seen that RG is primitive.

Finally, we shall show that KG is primitive for any field K. Let K' be a prime field. Since G satisfies $(*)$ and $|K'| \leq |G|$, we have already seen that $K'G$ is primitive. In view of Lemma 4.1, we need only show that $\Delta(G) = 1$.

Let g be a non-identity element in G. We can see that there exist infinite conjugate elements of g. In fact, if it is not true, then the set M of conjugate elements of g in G is a finite set. Since G satisfies $(*)$, for M, there exists $x_1, x_2 \in G$ such that M^{x_1} and M^{x_2} are mutually reduced. Since g is in M, $(x_1^{-1}gx_1)(x_2^{-1}fx_2)^{-1} \neq 1$ for any $f \in M$, and thus $x_1^{-1}gx_1 \neq x_2^{-1}fx_2$. Hence $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \neq f$ for all $f \in M$, which implies a contradiction $x^{-1}gx \notin M$, where $x = x_1x_2^{-1}$. This completes the proof of theorem. $\square$

We call the free product $A * B$ of two non-identity groups A and B a strict free product provided that it is not isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. In addition, we define a group G to be a locally strict free product if for each finite number of elements $g_1, \cdots, g_m$ in G, there exists a subgroup H of G which is isomorphic to a strict free product such that $\{g_1, \cdots, g_m\} \subset H$. The following corollary, which generalizes the result of [6], follows from Theorem 2.1:

Corollary 4.4. *Let R be a domain and G a locally strict free product. Suppose that G has a free subgroup whose cardinality is the same as that of G. If $|R| \leq |G|$ then the group ring RG is primitive.*

In particular, KG is primitive for any field K.

5 Proof of Theorem 2.2

Throughout this section, $F = \langle X \rangle$ denotes the free group with a base X. Let $G = \langle X \mid R \rangle$ denote the one relator group with the set of generators X with a relation R, where R is a cyclically reduced word in F. For a word W in F, if $R = W^n$, $n > 1$ and W is not a proper power in F, then G is called a one relator group with torsion. Let W be a word in F. We denote the normal closure of W in F by $\mathcal{N}_F(W)$. For a cyclically reduced word W, $\mathcal{W}_F(W)$ denotes the set of all cyclically reduced conjugates of both W and W^{-1}. If $W_i, \cdots, W_t$ are reduced words in F and $W = W_i \cdots W_t$ is also reduced, that is, there is no cancellation in forming the product $W_i \cdots W_t$, then we write $W \equiv W_i \cdots W_t$. For $Y \subset X$, $\langle Y \rangle_G$ is the subgroup of G generated by the homomorphic image in G of Y.

Lemma 5.1. *Let $n > 1$, and let $G = \langle X \mid R \rangle$, where W be a cyclically reduced word in F and $R = W^n$.*

(1) (See [18, Theorem], cf. [8]) *If $1 \neq V \in \mathcal{N}_F(R)$, then V contains a subword $S^{n-1}S_0$, where $S \equiv S_0 S_1 \in \mathcal{W}_F(W)$ and every generator which appears in W appears in S_0.*

(2) (See [12, Theorem]) *The centralizer of every non-trivial element in G is a cyclic group.*

Lemma 5.2. *For $n > 1$, let $G = \langle X \mid R \rangle$ with $|X| > 1$, where $R = W^n$ and W is a cyclically reduced word in F.*

(1) If $S, T \subseteq X$, then $\langle S \rangle_G \cap \langle T \rangle_G = \langle S \cap T \rangle_G$.

(2) $\Delta(G) = 1$.

Proof. (1): If $S \subseteq T$ or $T \subseteq S$, then the assertion is clear, and so we may assume $S \not\subseteq T$ and $T \not\subseteq S$. It is obvious that $\langle S \rangle_G \cap \langle T \rangle_G \supseteq \langle S \cap T \rangle_G$. Suppose, to the contrary, that $\langle S \rangle_G \cap \langle T \rangle_G \supsetneq \langle S \cap T \rangle_G$. Then there exist reduced words

$u = u(s, a, \cdots, b)$ in $\langle S\rangle \setminus \langle S\cap T\rangle$ and $v = v(t, c, \cdots, d)$ in $\langle T\rangle \setminus \langle S\cap T\rangle$ such that $uv \in \mathcal{N}_F(R)$, where $a, \cdots, b \in S$, $c, \cdots, d \in T$, $s \in S\setminus(S\cap T)$, and $t \in T\setminus(S\cap T)$. Let w be the reduced word for uv, say $w \equiv u_1v_1$, where $u \equiv u_1u_2$ and $v \equiv u_2^{-1}v_1$. Then $w \equiv u_1v_1 \in \mathcal{N}_F(R)$. However, u_1 involves s but not t, and v_1 involves t but not s, which contradicts the assertion of Lemma 5.1 (1).

(2): Suppose , to the contrary, $\Delta(G) \neq 1$; thus there exists $1 \neq g \in G$ such that $[G : C_G(g)] < \infty$. By Lemma 5.1 (2), $C_G(g)$ is cyclic and in fact infinite cyclic because $|G|$ is not finite. Thus G is virtually cyclic and so, as is well-known, there exists a normal subgroup N of finite order such that G/N is isomorphic to either the infinite cyclic group $\mathbb{Z}$ or the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ (See [9, 137p]).

Since a one relator group with torsion is isomorphic to neither $\mathbb{Z}$ nor $\mathbb{Z}_2 * \mathbb{Z}_2$, we may assume $N \neq 1$. In both cases of $G/N \simeq \mathbb{Z}$ and $G/N \simeq \mathbb{Z}_2 * \mathbb{Z}_2$, there exists $x \in G \setminus N$ such that $\langle x\rangle_G$ is a infinite cyclic subgroup of G. Since $|N|$ is finite, then it is easily seen that there exists $m > 0$ such that $x^{-m}fx^m = f$ for all $f \in N$, which implies $N \subset C_G(x^m)$; a contradiction, because a infinite cyclic group does not contain non-trivial finite subgroups. □

Let $X = \{x_1, x_2, \cdots, x_m\}$ with $m > 1$ and $F = \langle X\rangle$. To avoid unnecessary subscripts, we denote generators, $x_1, x_2, \cdots, x_m$, by $t, a, \cdots, b$. We consider the one relator group $G = \langle X\ |R\rangle$, where $R = W^n$, $n > 1$ and $W = W(t, a, \cdots, b)$ is a cyclically reduced word which is not a proper power. We assume that all generators appear in W. We shall see that there exists a normal subgroup L of G such that G/L is cyclic and L satisfies the assumption in Corollary 4.4. That is, G has the following type of subgroup G_∞ and L is a subgroup of it:

$$G_\infty = \langle X_\infty \mid R_i,\ i \in \mathbb{Z}\rangle \text{ with } R_i = W_i^n (n > 1), \tag{7}$$

where $X_\infty = \{a_j, \cdots, b_j \mid j \in \mathbb{Z}\}$ and for each $i \in \mathbb{Z}$, W_i is a cyclically reduced word in the free group $F_\infty = \langle X_\infty\rangle$. Let α_*, $\cdots$, β_* be respectively the minimum subscripts on a, $\cdots$, b occurring in W_0, and let $\alpha^*, \cdots, \beta^*$ be the maximum subscript on a, $\cdots$, b occurring in W_0, respectively. That is,

$$W_i = W_i(a_{\alpha_*+i}, \cdots, a_{\alpha^*+i}, \cdots, b_{\beta_*+i}, \cdots, b_{\beta^*+i}).$$

Let μ be the maximum number in $\{\alpha^* - \alpha_*, \cdots, \beta^* - \beta_*\}$. For $t \in \mathbb{Z}$, we set subgroups Q_t and P_t of G_∞ as follows:

$$\begin{cases} \text{For } \mu \neq 0, \\ Q_t \ = \langle a_{t+i}, \cdots, b_{t+j} \mid \alpha_* \leq i \leq \alpha^*,\ \cdots,\ \beta_* \leq j \leq \beta^*\rangle_{G_\infty}, \\ P_t \ = \langle a_{t+i}, \cdots, b_{t+j} \mid \alpha_* \leq i \leq \alpha^* - 1,\ \cdots,\ \beta_* \leq j \leq \beta^* - 1\rangle_{G_\infty}. \\ \text{For } \mu = 0, \\ Q_t \ = \langle a_{t+\alpha_*},\ \cdots,\ b_{t+\beta_*}\rangle_{G_\infty}, \\ P_t \ = 1. \end{cases} \tag{8}$$

Then P_t is a subgroup of Q_t and Q_t has the following presentation:

$$Q_t \simeq \langle a_{t+\alpha_*}, \cdots, a_{t+\alpha^*}, \ \cdots, \ b_{t+\beta_*}, \cdots, b_{t+\beta^*} \mid R_t \rangle. \tag{9}$$

In what follows, let $\nu = \beta^* - \beta_*$, and replacing the order of $a_i, \cdots, b_i$ in X_∞ if necessary, we may assume that $\mu = \alpha^* - \alpha_* \geq \cdots \geq \beta^* - \beta_* = \nu$. In view of the Magnus' method for Freiheitssatz, we may identify G_∞ as the union of the chain of the following G_i's (see [11] or [10]):

$$\begin{aligned} &G_\infty = \bigcup_{i=0}^{\infty} G_i, \ \text{where} \\ &G_0 = Q_0, \quad G_{2i} = Q_{-i} *_{P_{-i+1}} G_{2i-1}, \ \text{and} \ \ G_{2i+1} = G_{2i} *_{P_{i+1}} Q_{i+1}. \end{aligned} \tag{10}$$

By lemma 5.2 (1), we can get the next lemma:

Lemma 5.3. *If H is a subgroup of G_∞ generated by a finite subset Y of X_∞; namely $H = \langle Y \rangle_{G_\infty}$, then there exists a positive integer t such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.*

Lemma 5.4. *If G_∞ and W_i are as in (7), then for each finite number of elements $g_1, \cdots, g_m$ in G_∞, there exists an integer t such that $\langle g_1, \cdots, g_m, W_t \rangle_{G_\infty}$ is the free product $\langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_t \rangle_{G_\infty}$.*

Proof. Let Y be the subset of X_∞ consisting of generators appeared in g_i for all $i \in \{1, \cdots, m\}$. By virtue of Lemma 5.3, for $H = \langle Y \rangle_{G_\infty}$, there exists $t > 0$ such that $H \subseteq G_{2(t-1)}$ and $H \cap P_t = 1$.

Now, by (10), $G_{2t-1} = G_{2(t-1)} *_{P_t} Q_t$, where Q_t is as described in (9) and P_t is as described in (8). Since $W_t^n = R_t$ is the relator of Q_t, we have $\langle W_t \rangle_{G_\infty} \subset Q_t$. As is well known, $W_t^m \neq 1$ in Q_t for $1 \leq m < n$. Moreover, it holds that $P_t \cap \langle W_t \rangle_{Q_t} = 1$. In fact, if not so, there exists $m > 0$ such that $W_t^m \in P_t$ in Q_t. Since P_t is a free subgroup of Q_t by Freiheitssatz, we have that $1 \neq (W_t^m)^n = (W_t^n)^m$ in Q_t. However, this contradicts the fact that W_t^n is the relator of Q_t. We have thus shown that $P_t \cap \langle W_t \rangle_{Q_t} = 1$. Combining this with $H \cap P_t = 1$, we see that $\langle Y, W_t \rangle_{G_{2t-1}} = \langle Y \rangle_{G_{2t-1}} * \langle W_t \rangle_{G_{2t-1}} = H * \langle W_t \rangle_{G_\infty}$. Since $\langle g_1, \cdots, g_m \rangle_{G_\infty} \subseteq H$, we have that $\langle g_1, \cdots, g_m, W_t \rangle_{G_\infty} = \langle g_1, \cdots, g_m \rangle_{G_\infty} * \langle W_t \rangle_{G_\infty}$. □

Proof of Theorem 2.2 Let $G = \langle X \mid R \rangle$ be the one relator group with torsion, where $|X| > 1$, $R = W^n$, $n > 1$ and W is a cyclically reduced word which is not a proper power. If there exists $x \in X$ such that W contains none of x or x^{-1}, then G is a non-trivial free product of groups both of which are not isomorphic to $\mathbb{Z}_2$. Hence we may assume that $X = \{x_1, \cdots, x_m\}$ $(m > 1)$ and W contains either x_i or x_i^{-1} for all $i \in \{1, \cdots, m\}$. In this case, the cardinality of G is countable, and it is well-known that G has a non-cyclic free subgroup. Moreover, by Lemma 5.2 (2), we see that $\Delta(G) = 1$, and therefore, combining Corollary 4.4 with [19, Theorem 1], it suffices to show that there exists a normal subgroup L of G such that G/L is cyclic and L satisfies the following condition (C):

(C) For any $g_1, \cdots, g_l \in L$, there exists a free product $A * B$ in the set of subgroups of L such that $B \neq 1, a^2 \neq 1$ for some $a \in A$, and $g_1, \cdots, g_l \in A * B$.

There are now two cases to consider: whether or not the exponent sum $\sigma_x(W)$ of W on some generator x is zero.

If for each $x \in X$, $\sigma_x(W) \neq 0$, say $\sigma_{x_1}(W) = \alpha$ and $\sigma_{x_2}(W) = \beta$, then by the Magnus' method for Freiheitssatz, $G \simeq \langle a^\beta, x_2, \cdots, x_m \mid R^* \rangle \subset E$, where $R^* = (W^*)^n$, $W^* = W^*(a^\beta, x_2, \cdots, x_m)$ and $E = \langle a, x_2, \cdots, x_m \mid R^* \rangle$. Let $N = \mathcal{N}_{F_*}(x_2 a^\alpha, x_3 \cdots, x_m)$, where $F_* = \langle a, x_2, \cdots, x_m \rangle$. Then we have that $N \supset \mathcal{N}_{F_*}(R^*)$ and $N/\mathcal{N}_{F_*}(R^*) \simeq G_\infty$, where G_∞ is as in (7), and so we may let $G_\infty = N/\mathcal{N}_{F_*}(R^*)$.

Let $F_G = \langle a^\beta, x_2, \cdots, x_m \rangle$ and $L = (N \cap F_G)/\mathcal{N}_{F_G}(R^*)$. Then we can easily see that L can be isomorphically embedded in G_∞ and that G is a cyclic extension of L.

Let $g_1, \cdots, g_l$ $(l > 0)$ be in L with $g_i \neq 1$. In case of $n > 2$, since $L \subset G_\infty$, by Lemma 5.4, there exists $t > 0$ such that $\langle g_1, \cdots, g_l \rangle_{G_\infty} * \langle W_t^* \rangle_{G_\infty}$. We have then that $1 \neq W_t^* \in L$ and $(W_t^*)^2 \neq 0$ because $n > 2$, and so L satisfies the condition (C). On the other hand, in case of $n = 2$, let $p > 0$ be the maximum number such that either $a^{p\beta}$ or $a^{-p\beta}$ is appeared in $W^* = W^*(a^\beta, x_2, \cdots, x_m)$. Set $v = a^{(p+1)\beta} x_2 a^{-(p+1)\beta} x_2^{-1}$ so that $v \in F_G$. Moreover, since $\sigma_a(v) = 0$ and $\sigma_{x_2}(v) = 0$, the homomorphic image $\overline{v}$ of v is contained in L. Suppose that $\overline{v}^2 = 1$; namely, $v^2 \in \mathcal{N}_{F_G}(R^*)$. In view of Lemma 5.2 (1), a reduced word v^2 contains a subword $S_0 S_1 S_0$ such that $S_0 S_1$ is a cyclic shift of W^* and S_0 contains all generators appeared in W^*. Since only two letters a and x_2 are appeared in v^2, we have that $W^* = W^*(a^\beta, x_2)$. Moreover, $S_0 S_1 S_0$ involves a subword of type $x_2^{\varepsilon_1} a^q x_2^{\varepsilon_2}$ with $|q| \leq |p\beta|$, where $\varepsilon_i = \pm 1$. However, since $|(p+1)\beta| > |q|$, there exists no such subword in v^2, which implies a contradiction. We have thus shown that $\overline{v}^2 \neq 1$. By virtue of Lemma 5.4, for $g_1, \cdots, g_l$ and $\overline{v}$, there exists $t > 0$ such that $\langle \overline{v}, g_1, \cdots, g_l \rangle_{G_\infty} * \langle W_t^* \rangle_{G_\infty}$. Since $1 \neq W_t^* \in L$ and $\overline{v}^2 \neq 1$, we have thus proved that L satisfies the condition (C).

If W has a zero exponent sum $\sigma_x(W)$ on x for some $x \in X$, say $\sigma_{x_1}(W) = 0$, then we set $N = \mathcal{N}_F(x_2, x_3 \cdots, x_m)$ and $L = N/\mathcal{N}_F(R)$, where $F = \langle x_1, x_2, \cdots, x_m \rangle$, $R = W^n$ and $W = W(x_1, \cdots, x_m)$. It is obvious that $L \simeq G_\infty$ and G is a cyclic extension of L. Moreover, we can easily see that L satisfies the condition (C). This completes the proof of the theorem.

References

[1] B. O. Balogun, *On the primitivity of group rings of amalgamated free products*, Proc. Amer. Math. Soc., 106(1)(1989), 43-47

[2] M. A. Chaudhry, M. J. Crabb and M. McGregor, *The primitivity of semigroup algebras of free products* Semigroup Forum, **54**(2)(1997), 221-229

[3] O. I. Domanov, *A prime but not primitive regular ring*, Uspekhi. Mat. Nauk., **32)**(1977), 219-220

[4] O. I. Domanov, *Primitive group algebras of polycyclic groups*, Sibirsk. Mat. Ž., **19(1)**(1978), 37-43

[5] D. R. Farkas and D. S. Passman, *Primitive Noetherian group rings*, Comm. Algebra, **6(3)**(1978), 301-315.

[6] E. Formanek, *Group rings of free products are primitive*, J. Algebra, **26**(1973), 508-511

[7] E. Formanek and R. L. Snider, *Primitive group rings*, Proc. Amer. Math. Soc., **36**(1972), 357-360

[8] J. Howie and S. J. Pride, *A spelling theorem for staggered generalized 2-complexes, with applications*, Invent. math., 76 (1984), 55-74

[9] D. Juan-Pineda and Ian J. Leary, *On classifying spaces for the family of virtually cyclic subgroups*, Recent developments in algebraic topology, Contemp. Math., 407, Amer. Math. Soc., Providence, RI, 407(2006), 135-145

[10] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergeb. der Math. und ihrer Grenzgeb., 89, Springer, Berlin 1977

[11] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Wiley 1966, Second Edition, Dover Publication, New York, 1976

[12] B. B. Newman, *The soluble subgroups of a one-relator group with torsion*, J. Austral. Math. Soc., 16(1973), 278-285

[13] T. Nishinaka, *Group rings of proper ascending HNN extensions of countably infinite free groups are primitive*, J. Algebra, **317**(2007), 581-592

[14] T. Nishinaka, *Group rings of countable non-abelian locally free groups are primitive*, Int. J. algebra and computation, **21**(3) (2011), 409-431

[15] A. Ju. Ol'shanskiĭ, *An infinite simple torsion-free Noetherian group*, Izv. Akad. Nauk BSSR, Ser. Mat., 43(1979), 1328-1393.

[16] D. S. Passman, *Primitive group rings*, Pac. J. Math., **47**(1973), 499-506.

[17] J. E. Roseblade, *Prime ideals in group rings of polycyclic groups*, Proc. London Math. Soc., **36(3)**(1978), 385-447. Corrigenda "Prime ideals in group rings of polycyclic groups" Proc. London Math. Soc., **36(3)**(1979), 216-218.

[18] Paul E. Schupp, *A Strengthened Freiheitssatz*, Math. Ann., 221(1976), 73-80.

[19] A. E. Zalesskii, *The group algebras of solvable groups*, Izv. Akad. Nauk BSSR, ser Fiz. Mat., (1970), 13-21.