MULTIPLICATIVE SETS OF IDEMPOTENTS IN A SEMILOCAL RING

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An element \( e \) of a ring \( R \) is called an idempotent if \( e^2 = e \). An idempotent \( e \) is said to be primitive if there are no two non-zero idempotent \( f, g \in R \) such that \( e = f + g \) and \( fg = gf = 0 \).

**Proposition 1.** Let \( K \) be a field of characteristic \( p \neq 2 \). Let \( R \) be a \( K \)-subalgebra of the ring \( M_n(K) \) of \( n \times n \) matrices over \( K \) containing matrix units \( e_{11}, e_{22}, \ldots, e_{nn} \). Let \( M \) denote the set consisting of primitive idempotents and \( 0 \). Suppose that, for any \( e, f \in M \), \( ef \) is either an idempotent or a nilpotent element. Then \( R \) is isomorphic to a \( K \)-subalgebra of the ring \( T_n(K) \) of all upper triangular matrices over \( K \).

**Proof.** Assume that \( e_{ij}, e_{ji} \in R \) for some \( i \neq j \). Then \( R \) contain two primitive idempotents \( e = e_{ii} + e_{ij} \) and \( f = e_{ii} + e_{ji} \). We see that \( ef = 2e_{ii} \). Since \( \text{char}(K) \neq 2 \), \( 2e_{ii} \) is neither an idempotent nor a nilpotent element. Hence, if \( e_{ij} \in R \) for some \( i \neq j \), then \( e_{ji} \not\in R \). Now we define an order on the set \( \{1, 2, \ldots, n\} \). If \( e_{ij} \in R \), then we define \( i \leq j \). Since \( e_{ii} \in R \) for all \( i \in \{1, 2, \ldots, n\} \), we have \( i \leq i \). If \( i \leq j \) and \( j \leq k \), then \( e_{ij}, e_{jk} \in R \), and hence \( e_{ik} = e_{ij}e_{jk} \in R \). Therefore \( i \leq k \). If \( i \leq j \) and \( j \leq i \), then \( e_{ij}, e_{ji} \in R \). As we saw in the first paragraph of the proof, \( i = j \) in this case. Therefore \( \leq \) is a partial order on \( \{1, 2, \ldots, n\} \). Let \( m \) be a minimal element of the ordered set \( \{1, 2, \ldots, n\} \). Then \( e_{mj} \not\in R \) for any \( j \neq m \). Renumbering the elements in \( \{1, 2, \ldots, n\} \), we may assume that \( m = 1 \).

Then we see \( R \subset e_{11}K + (e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn}) \). Using induction on \( n, (e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn}) \) is isomorphic to a \( K \)-subalgebra of the ring \( T_{n-1}(K) \). Hence \( R \) is isomorphic to a \( K \)-subalgebra of \( T_n(K) \).

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The following example show that the proposition above is not true when the field $K$ is of characteristic 2.

**Example 1.** Consider the ring $R = M_2(GF(2))$ of $2 \times 2$ matrices over the Galois field $GF(2)$ and let $M$ denote the set consisting of all primitive idempotents in $R$ and zero. We can easily see that for any $e, f \in M$, $ef$ is either an idempotent or a nilpotent element.

Next, we prove the following.

**Proposition 2.** Let $e$ be a primitive idempotent of a ring $R$. If $ef$ is a non-zero idempotent of $R$ for some element $f \in R$, then $ef$ is a primitive idempotent.

**Proof.** Assume that $ef = a + b$ for some orthogonal idempotents $a, b \in R$. Then $a + b = ef = ea + eb$, and so $a = (a + b)a = (ea + eb)a = ca$. Similarly, we have $b = eb$. We can easily see that $e - ae$ and $ae$ are orthogonal idempotents and $e = (e - ae) + ae$. Since $e$ is a primitive idempotent, either $e = ae$ or $ae = 0$ holds. If $e = ae$, then $b = eb = aceb = ab = 0$. On the other hand, if $e = ae$, then $a = a^2 = aea = 0$. This proves that $ef$ is primitive.

Let $R$ be a ring. Let $M$ and $E$ denote the set consisting of all primitive idempotents in $R$ and zero and the set of idempotents in $R$, respectively. If $S$ is a multiplicatively closed set of idempotents in $R$ containing 0, then $M \cap S$ is also multiplicatively closed.

By Zorn’s lemma, we have the following.

**Proposition 3.** Every multiplicatively closed subset of $M$ (resp. $E$) is contained in a maximal multiplicatively closed subset of $M$ (resp. $E$).

**Example 2.** Let $M_2(K)$ be a ring of $2 \times 2$ matrices over a field $K$. We can see that $(e_{11} + e_{12}K) \cup \{0\}$ is a maximal multiplicatively closed subset of $M$.

**Theorem 1.** Let $R$ be a ring and let $M$ denote the set consisting of all primitive idempotents in $R$ and zero. Suppose that there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of $R$ such that $1 = e_1 + e_2 + \cdots + e_n$. Then $\{0, e_1, e_2, \cdots, e_n\}$ is a maximal multiplicatively closed set in $M$.

**Proof.** Suppose, on the contrary, that there is a multiplicativity closed subset $G$ of $M$ which properly contains $\{0, e_1, e_2, \cdots, e_n\}$ and let $f \in G \setminus \{0, e_1, e_2, \cdots, e_n\}$. Since $e_1f e_2$ is a nilpotent element, $e_1f e_2$ must be 0. Similarly we have $e_1f e_i = 0$ for $i = 3, \ldots, n$. Hence we have $e_1f(1 - e_1) = e_1f e_2 + \cdots + e_1f e_n = 0$. Similarly we have $(1 - e_1)f e_1 = 0$. Therefore $e_1f = e_1f e_1 = fe_1$, that is $e_1$ and $f$ are commutative.
By the same way, we can see that \( f \) and \( e_i \) are commutative for \( i = 2, \ldots, n \).
Now we can easily see that \( e_1 f, e_2 f, \ldots, e_n f \) are primitive orthogonal idempotents.
Since \( 1 = e_1 f + \cdots + e_n f \) and since \( f \) is primitive, we conclude that \( f = e_i f \) for \( i \).
Since \( f \) and \( e_1 \) are commutative, \( e_1 f \) and \( e_1 (1 - f) \) are orthogonal idempotents.
Since \( e_1 = e_1 f + e_1 (1 - f) \) and since \( f \) is primitive, we see \( e_1 (1 - f) = 0 \). Then \( e_1 = e_1 f = f \), a contradiction.

**Example 3.** Consider the ring \( R = \mathbb{Z} + M_2(\mathbb{Q}[x]) \). \( R \) is an order of \( M_2(\mathbb{Q}[x]) \).
We can easily see that the idempotents of \( R \) are only 0 and 1.

**Theorem 2.** Let \( R \) be a ring and let \( M \) denote the set consisting of all primitive idempotents in \( R \) and zero. Suppose that 1 is a sum of primitive orthogonal idempotents. Then \( M \) is closed under multiplication if and only if \( R \) is a direct sum of rings with no non-trivial idempotents.

**Proof.** Suppose that \( M \) is closed under multiplication and that there are primitive orthogonal idempotents \( e_1, e_2, \ldots, e_n \) of \( R \) such that \( 1 = e_1 + e_2 + \cdots + e_n \).
Since \( \{0, e_1, e_2, \cdots, e_n\} \) is a maximal multiplicatively closed set in \( M \) by Theorem 1, we conclude that \( M = \{0, e_1, e_2, \cdots, e_n\} \). Then \( e_1, e_2, \cdots, e_n \) are central orthogonal idempotents and \( R = e_1 R \oplus \cdots \oplus e_n R \). Since each \( e_i \) is primitive, each \( e_i R \) has no non-trivial idempotents.
In [2], D. Dolžan proved that \( M \) is closed under multiplication if and only if \( R \) is a direct sum of local rings ([2, Corollary 5.6]). Now we generalize this result to semiperfect rings. Let \( R \) denote a ring and \( J \) denote its Jacobson radical. A ring \( R \) is called semiperfect if \( R \) is semilocal and idempotents of \( R/J \) can be lifted to \( R \). All basic results concerning rings can be found in [1].
If \( R \) be a semiperfect ring, then there are primitive orthogonal idempotents \( e_1, e_2, \cdots, e_n \) of \( R \) such that \( 1 = e_1 + e_2 + \cdots + e_n \) and each \( e_i R e_i \) is a local ring.
Hence we have the following.

**Corollary 1.** Let \( R \) be a semiperfect ring and \( M \) be the set of all minimal idempotents and zero in \( R \). Then the set \( M \) is closed under multiplication if and only if \( R \) is a direct sum of local rings.

Let \([M]\) denote the set \( \{eR \mid e \in M\} \), that is, \([M]\) is the set of right ideals of the form \( eR \) for some primitive idempotent \( e \) and the ideal 0.

**Theorem 3.** Let \( R \) be a semiperfect ring and \([M]\) be the set of right ideals of the form \( eR \) for some primitive idempotent \( e \) and the ideal 0. Then the set \([M]\) is closed under multiplication if and only if \( R \) is a finite direct sum of matrix rings over some local ring.
Proof. If $R$ is a finite direct sum of matrix ring over some local ring, then clearly $M$ is closed under multiplication. Let $e$ and $f$ be two primitive idempotents of $R$. Then either $eRfR = 0$ or $eRfR = gR$ for some primitive idempotent $g \in R$. If $eRfR = 0$, then $(fReR)^2 = 0$. In this case $fReR$ is not a nonzero direct summand of $R$, and so we conclude that $fReR = 0$. If $eRfR = gR$ for some primitive idempotent $g \in R$, then $eR \supseteq gR$. Using modular law, we have $eR = eR \cap (gR \oplus (1 - g)R) = gR \oplus eR \cap (1 - g)R$. Since $eR$ is indecomposable, we conclude that $gR = eR$. Thus $eRfR = eR$, and so $eRfRe = eRe$. Then we can write $e = \sum_{i=1}^{n} ea_i fb_i e$ for some $a_i, b_i \in R$. Since $eRe$ is a local ring, for some $k$, $ea_k fb_k e$ is invertible in $eRe$. Similarly there exists $c, d \in R$ such that $fcedf$ is invertible in $fRf$. These mean that $eR \cong fR$. Since $R$ is semiperfect, $R = e_1 R \oplus \cdots \oplus e_n R$ for some primitive idempotents $e_1, \cdots, e_n$. By the fact proved above, $R = R_1 \oplus \cdots \oplus R_m$ such that each two-sided ideal $R_i$ is a finite direct sum of isomorphic indecomposable models. Then $R \cong \text{End}(R_1) \oplus \cdots \oplus \text{End}(R_m)$. Thus each $R_i \cong \text{End}(R_i)$ is a matrix ring over a local ring.

References