MULTIPLICATIVE SETS OF IDEMPOTENTS IN A SEMILOCAL RING

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An element e of a ring R is called an idempotent if $e^2 = e$. An idempotent e is said to be primitive if there are no two non-zero idempotent $f,\ g \in R$ such that $e = f + g$ and $fg = gf = 0$.

Proposition 1. *Let K be a field of charactristic $p \neq 2$. Let R be a K-subalgebra of the ring $M_n(K)$ of $n \times n$ matrices over K containing matrix units $e_{11}, e_{22}, \cdots, e_{nn}$. Let M denote the set consisting of primitive idempotents and 0. Suppose that, for any $e,\ f \in M$, ef is either an idempotent or a nilpotent element. Then R is isomorphic to a K-subalgebra of the ring $T_n(K)$ of all upper triangular matrices over K.*

Proof. Assume that $e_{ij}, e_{ji} \in R$ for some $i \neq j$. Then R contain two primitive idempotents $e = e_{ii} + e_{ij}$ and $f = e_{ii} + e_{ji}$. We see that $ef = 2e_{ii}$. Since $char(K) \neq 2$, $2e_{ii}$ is neither an idempotent nor a nilpotent element. Hence, if $e_{ij} \in R$ for some $i \neq j$, then $e_{ji} \notin R$. Now we define an order on the set $\{1, 2, \cdots, n\}$. If $e_{ij} \in R$, then we define $i \leq j$. Since $e_{ii} \in R$ for all $i \in \{1, 2, \cdots, n\}$, we have $i \leq i$. If $i \leq j$ and $j \leq k$, then $e_{ij},\ e_{jk} \in R$, and hence $e_{ik} = e_{ij}e_{jk} \in R$. Therefore $i \leq k$. If $i \leq j$ and $j \leq i$, then $e_{ij},\ e_{ji} \in R$. As we saw in the first paragraph of the proof, $i = j$ in this case. Therefore $\leq$ is a partial order on $\{1, 2,\ \cdots, n\}$. Let m be a minimal element of the ordered set $\{1, 2,\ \cdots, n\}$. Then $e_{mj} \notin R$ for any $j \neq m$. Renumbering the elements in $\{1, 2,\ \cdots, n\}$, we may assume that $m = 1$. Then we see $R \subset e_{11}K + (e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn})$. Using induction on n, $(e_{22} + \cdots + e_{nn})R(e_{22} + \cdots + e_{nn})$ is isomorphic to a K-subalgebra of the ring $T_{n-1}(K)$. Hence R is isomorphic to a K-subalgebra of $T_n(K)$.

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The following example show that the proposition above is not true when the field K is of characteristic 2.

Example 1. *Consider the ring* $R = M_2(GF(2))$ *of* 2×2 *matrices over the Galois field* $GF(2)$ *and let* M *denote the set consisting of all primitive idempotents in* R *and zero. We can easily see that for any* $e,\ f \in M$*,* ef *is either an idempotent or a nilpotent element.*

Next, we prove the following.

Proposition 2. *Let* e *be a primitive idempotent of a ring* R*. If* ef *is a non-zero idempotent of* R *for some element* $f \in R$*, then* ef *is a primitive idempotent.*

Proof. Assume that $ef = a + b$ for some orthogonal idempotents $a,\ b \in R$. Then $a + b = ef = ea + eb$, and so $a = (a + b)a = (ea + eb)a = ea$. Similarly, we have $b = eb$. We can easily see that $e - ae$ and ae are orthogonal idempotents and $e = (e - ae) + ae$. Since e is a primitive idempotent, either $e = ae$ or $ae = 0$ holds. If $e = ae$, then $b = eb = aeb = ab = 0$. On the other hand, if $e = ae$, then $a = a^2 = aea = 0$. This proves that ef is primitive.

Let R be a ring. Let M and E denote the set consisting of all primitive idempotents in R and zero and the set of idempotents in R, respectively. If S is a multiplicatively closed set of idempotents in R containing 0, then $M \cap S$ is also multiplicatively closed.

By Zorn's lemma, we have the following.

Proposition 3. *Every multiplicatively closed subset of* M *(resp.* E*) is contained in a maximal multiplicatively closed subset of* M *(resp.* E*).*

Example 2. *Let* $M_2(K)$ *be a ring of* 2×2 *matrices over a field* K*. We can see that* $(e_{11} + e_{12}K) \cup \{0\}$ *is a maximal multiplicatively closed subset of* M*.*

Theorem 1. *Let* R *be a ring and let* M *denote the set consisting of all primitive idempotents in* R *and zero. Suppose that there are primitive orthogonal idempotents* $e_1, e_2, \cdots, e_n$ *of* R *such that* $1 = e_1 + e_2 + \cdots + e_n$*. Then* $\{0, e_1, e_2, \cdots, e_n\}$ *is a maximal multiplicatively cosed set in* M*.*

Proof. Suppose, on the contrary, that there is a multiplicativery colsed subset G of M which properly contains $\{0, e_1, e_2, \cdots, e_n\}$ and let $f \in G \setminus \{0, e_1, e_2, \cdots, e_n\}$. Since $e_1 f e_2$ is a nilpotent element, $e_1 f e_2$ must be 0. Similarly we have $e_1 f e_i = 0$ for $i = 3, \ldots, n$. Hence we have $e_1 f(1 - e_1) = e_1 f e_2 + \cdots + e_1 f e_n = 0$. Similarly we have $(1 - e_1) f e_1 = 0$. Therefore $e_1 f = e_1 f e_1 = f e_1$, that is e_1 and f are commutative.

By the same way, we can see that f and e_i are commutative for $i = 2, \cdots, n$. Now we can easily see that $e_1 f, e_2 f, \cdots, e_n f$ are primitive orthogonal idempotents. Since $1 = e_1 f + \cdots +_n f$ and since f is primitive, we conclude that $f = e_i f$ for i. Since f and e_1 are commutative, $e_1 f$ and $e_1(1-f)$ are orthogonal idempotents. Since $e_1 = e_1 f + e_1(1-f)$ and since f is primitive, we see $e_1(1-f) = 0$. Then $e_1 = e_1 f = f$, a contradiction.

Example 3. *Consider the ring $R = \mathbf{Z} + M_2(\mathbf{Q}[x]x)$. R is an order of $M_2(\mathbf{Q}[x])$. We can easily see that the idempotents of R are only 0 and 1.*

Theorem 2. *Let R be a ring and let M denote the set consisting of all primitive idempotents in R and zero. Suppose that 1 is a sum of primitive orthogonal idempotents. Then M is closed under multiplication if and only if R is a direct sum of rings with no non-trivial idempotents.*

Proof. Suppose that M is closed under multiplication and that there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of R such that $1 = e_1 + e_2 + \cdots + e_n$. Since $\{0, e_1, e_2, \cdots, e_n\}$ is a maximal multiplicatively cosed set in M by Theorem 1, we conclude that $M = \{0, e_1, e_2, \cdots, e_n\}$. Then $e_1, e_2, \cdots, e_n$ are central orthogonal idempotents and $R = e_1 R \oplus \cdots \oplus e_n R$. Since each e_i is primitive, each $e_i R$ has no non-trivial idempotents.

In [2], D. Dolžan proved that M is closed under multiplication if and only if R is a direct sum of local rings([2, Corollary 5.6]). Now we generalize this result to semiperfect rings. Let R denote a ring and J denote its Jacobson radical. A ring R is called semiperfect if R is semilocal and idempotents of R/J can be lifted to R. All basic results concerning rings can be found in [1].

If R be a semiperfect ring, then there are primitive orthogonal idempotents $e_1, e_2, \cdots, e_n$ of R such that $1 = e_1 + e_2 + \cdots + e_n$ and each $e_i R e_i$ is a local ring. Hence we have the following.

Corollary 1. *Let R be a semiperfect ring and M be the set of all minimal idempotents and zero in R. Then the set M is closed under multiplication if and only if R is a direct sum of local rings.*

Let $[M]$ denote the set $\{eR \mid e \in M\}$, that is, $[M]$ is the set of right ideals of the form eR for some primitive idempotent e and the ideal 0.

Theorem 3. *Let R be a semiperfect ring and $[M]$ be the set of right ideals of the form eR for some primitive idempotent e and the ideal 0. Then the set $[M]$ is closed under multiplication if and only if R is a finite direct sum of matrix rings over some local ring.*

Proof. If R is a finite direct sum of matrix ring over some local ring, then clearly M is closed under multiplication. Let e and f be two primitive idempotents of R. Then either $eRfR = 0$ or $eRfR = gR$ for some primitive idempotent $g \in R$. If $eRfR = 0$, then $(fReR)^2 = 0$. In this case $fReR$ is not a nonzero direct summand of R, and so we conclude that $fReR = 0$. If $eRfR = gR$ for some primitive idempotent $g \in R$, then $eR \supseteq gR$. Using modular law, we have $eR = eR \cap (gR \oplus (1-g)R) = gR \oplus eR \cap (1-g)R$. Since eR is indecomposable, we conclude that $gR = eR$. Thus $eRfR = eR$, and so $eRfRe = eRe$. Then we can write $e = \sum_{i=1}^{n} ea_i fb_i e$ for some $a_i,\ b_i \in R$. Since eRe is a local ring, for some k, $ea_k fb_k e$ is invertible in eRe. Similarly there exists $c,\ d \in R$ such that $fcedf$ is invertible in fRf. These mean that $eR \cong fR$. Since R is semiperfect, $R = e_1 R \oplus \cdots \oplus e_n R$ for some primitive idempotents $e_1, \cdots, e_n$. By the fact proved above, $R = R_1 \oplus \cdots \oplus R_m$ such that each two-sided ideal R_i is a finite direct sum of isomorphic indecomposable modeles. Then $R \cong End(R_1) \oplus \cdots \oplus End(R_m)$. Thus each $R_i \cong End(R_i)$ is a matrix ring ove a local ring.

References

[1] F. W. Anderson and K. R. Fuller: Rings and Categories of Modules, Second Edition, Springer-Verlag, New York-Heidelberg-Berlin, 1992.

[2] Dolžan, D.: Multiplicative sets of idempotents in a finite ring, J. Algebra 304 (2006), no. 1, 271–277.